PCTM 2012

Kathleen Acker

Robert McGee

Carol Serotta

Connecting Algebra, Geometry, Number Theory, and the History of Mathematics

In this talk we hope to highlight connections between common curricular topics by viewing them through historical lenses.

We hope that seeing these relationships on different levels will capture the imagination of the students (and teachers!)

Archimedes *Palimpsest*

Archimedes *Palimpsest*

Damaged manuscript, mold, ink and paint

Missing Pages; Painted Images

High tech computer photography work, e.g., Photoshop and spectrum Analysis

Community of scholars formed: mathematicians, imagers, physicists, linguists, archival experts, restoration experts

Archimedes *Palimpsest*

Potential Infinity

Very large (or small) but essentially finite values, chosen for convenience.

Actual Infinity

The number of points on a line.

Proofs done by Ancient Greek Mathematicians made use of potential infinity, but avoided use of actual infinity.

The discussion of infinity can occur in both pre-calculus and calculus courses.

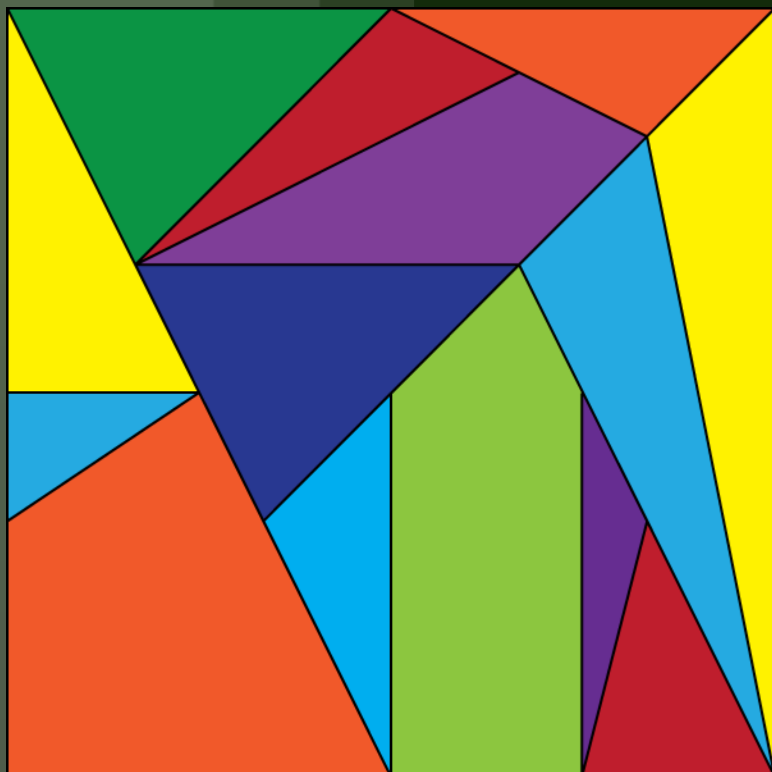
Stomachion

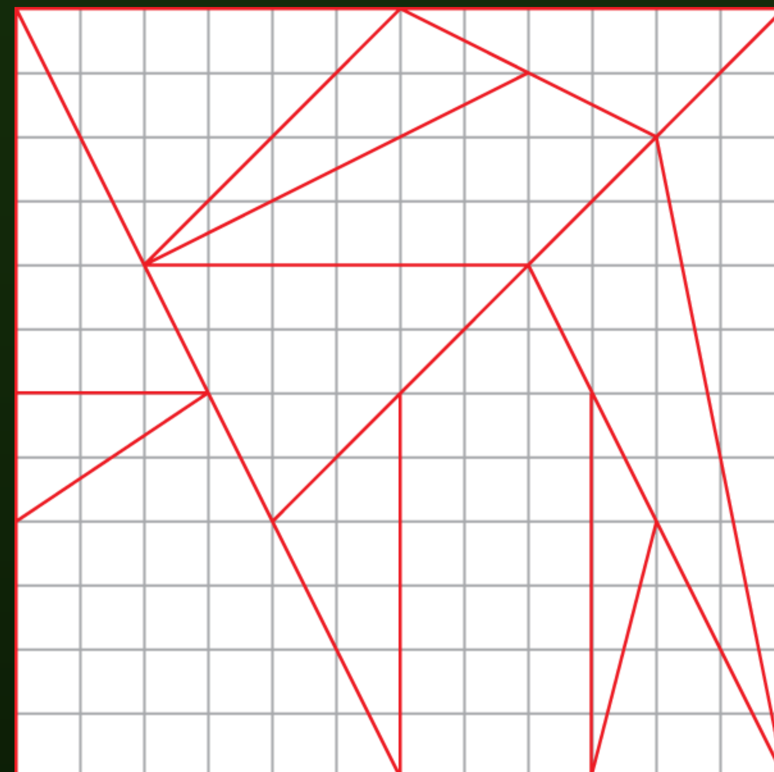
Is a 14 piece puzzle also know as Archimedes Box

Assembled it creates a square.

There are 536 distinct solutions. Allowing for rotations and reflections, the number grows to 17,152.

Stomachion





The grid is 12 x 12.

The vertices of each piece lie on a vertex, making lattice polygons.

Pick's Theorem

Pick's Theorem is a method of determining the area of a polygon in a plane.

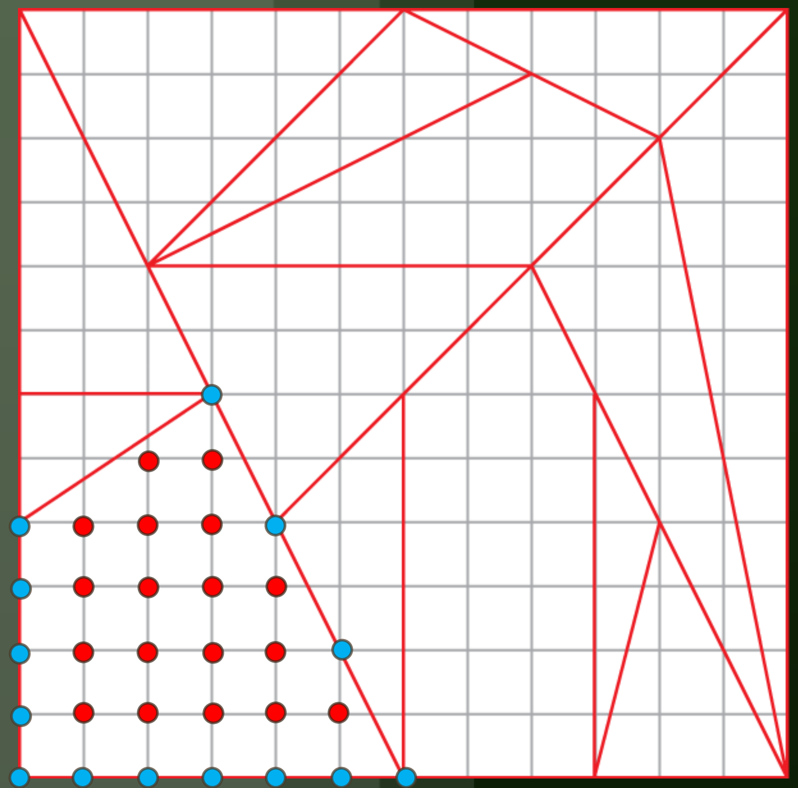
Let:

L = number of lattices points internal to the lattice polygon

B = number of lattice points on the boundary of the lattice polygon

$$Area = L + \frac{B}{2} - 1$$

Pick's Theorem



Boundary points, B, are noted in blue.

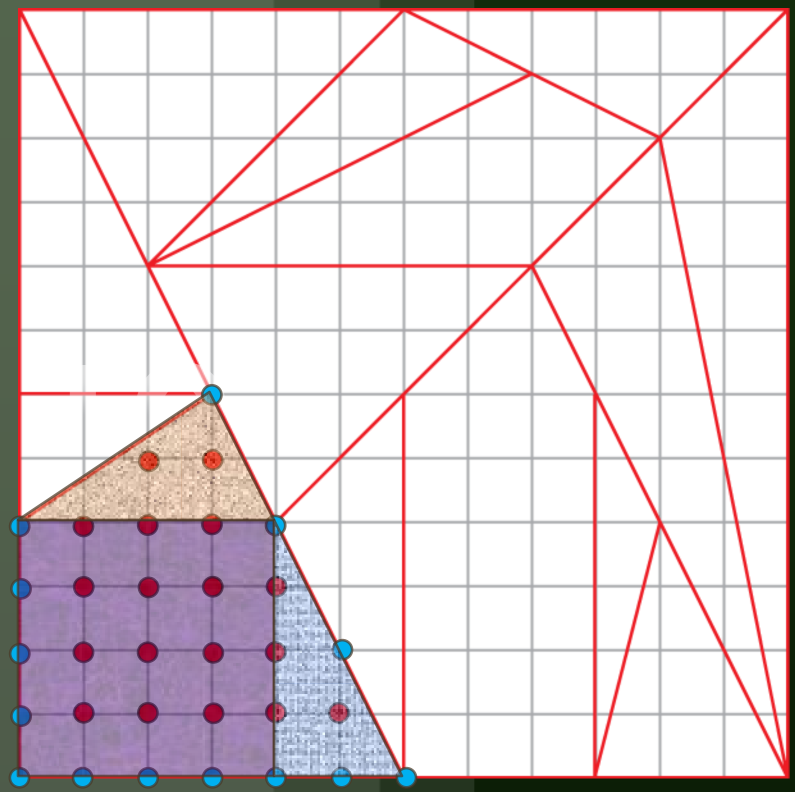
Internal points, L, are noted in red.

$$Area = L + \frac{B}{2} - 1$$

$$= 18 + \frac{14}{2} - 1$$

$$= 24$$

Pick’s Theorem



To Check

Area of Square is 16.

Area of brown
Triangle is 4

Area of blue triangle is
4

16+4+4 = 24

Geometry to Algebra

Geometry and algebra are separate courses in the secondary curriculum, so making the connections between the topics can be a challenge.

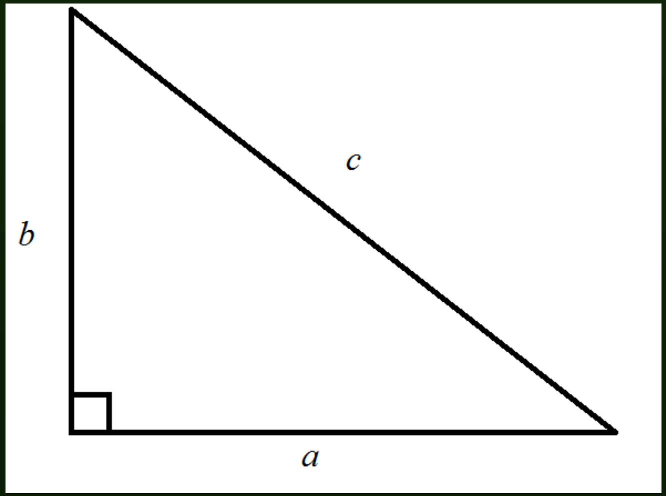
Function theory and technology nicely connect algebra and geometry.

Are there other connections between geometry and algebra?

Pythagorean Theorem

For a right triangle with legs a and b, and hypotenuse c

$$a^2 + b^2 = c^2$$



Euclid’s Pythagorean Theorem

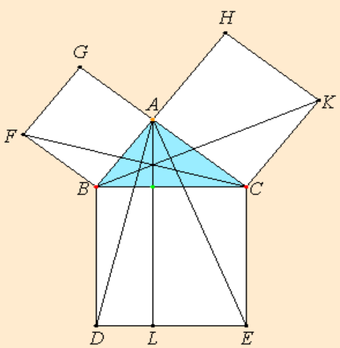
Euclid’s Elements
Book I
Proposition 47

In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

Let *ABC* be a right-angled triangle having the angle *BAC* right.

I say that the square on *BC* equals the sum of the squares on *BA* and *AC*.

Describe the square *BDEC* on *BC*, and the squares *GB* and *HC* on *BA* and *AC*. Draw *AL* through *A* parallel to either *BD* or *CE*, and join *AD* and *FC*. I.46 I.31 Post.1



Since each of the angles *BAC* and *BAG* is right, it follows that with a straight line *BA*, and at the point *A* on it, the two straight lines *AC* and *AG* not lying on the same side make the adjacent angles equal to two right angles, therefore *CA* is in a straight line with *AG*. I.Def.22 I.14

For the same reason *BA* is also in a straight line with *AH*.

Since the angle *DBC* equals the angle *FBA*, for each is right, add the angle *ABC* to each, therefore the whole angle *DBA* equals the whole angle *FBC*. I.Def.22 Post.4 C.N.2

Since *DB* equals *BC*, and *FB* equals *BA*, the two sides *AB* and *BD* equal the two sides *FB* and *BC* respectively, and the angle *ABD* equals the angle *FBC*, therefore the base *AD* equals the base *FC*, and the triangle *ABD* equals the triangle *FBC*. I.Def.22 I.4

David Joyce Clark University

Euclid's Pythagorean Theorem

To understand Euclid's Proof, we need some explanation.

Let's look at an animation.

animation 1 v5.swf

Euclid's Pythagorean Theorem

Please watch Euclid's proof of the Pythagorean Theorem.

Brides Chair Pythagorous v2 .avi

Another Pythagorean Animation

Watch:

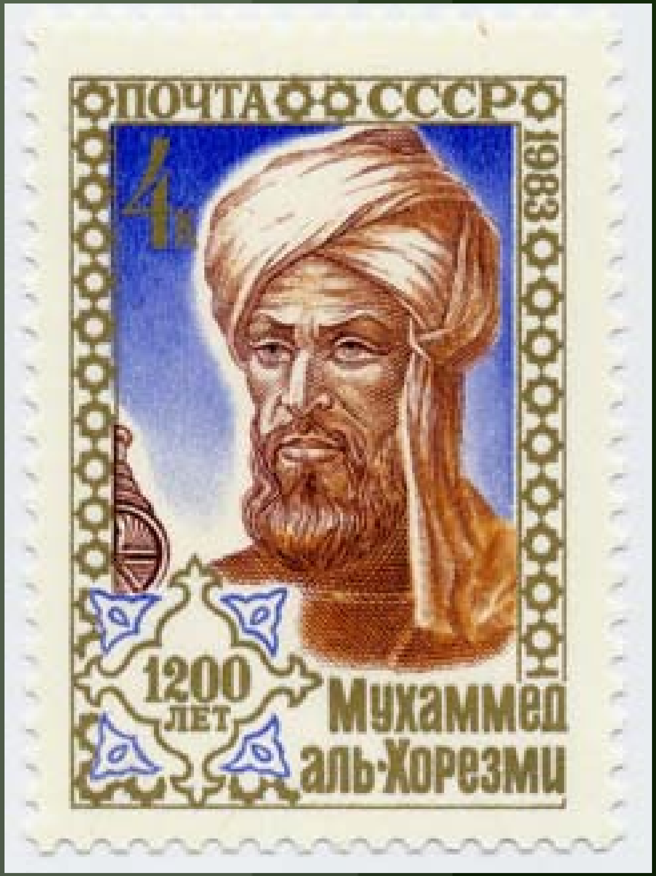
pyththm3.swf

Another Pythagorean Animation

Watch:

pyththm4.swf

Completing the Square



Abū ʿAbdallāh Muḥammad ibn Mūsā al-Khwārizmī wrote

The Compendious Book on Calculation by Completion and Balancing (825 AD)

Let's look at one of the problems.

Completing the Square

One square, and ten roots of the same, are equal to thirty-nine *dirhems.* That is to say, what must be the square which, when increased by ten of its own roots, amounts to 39?

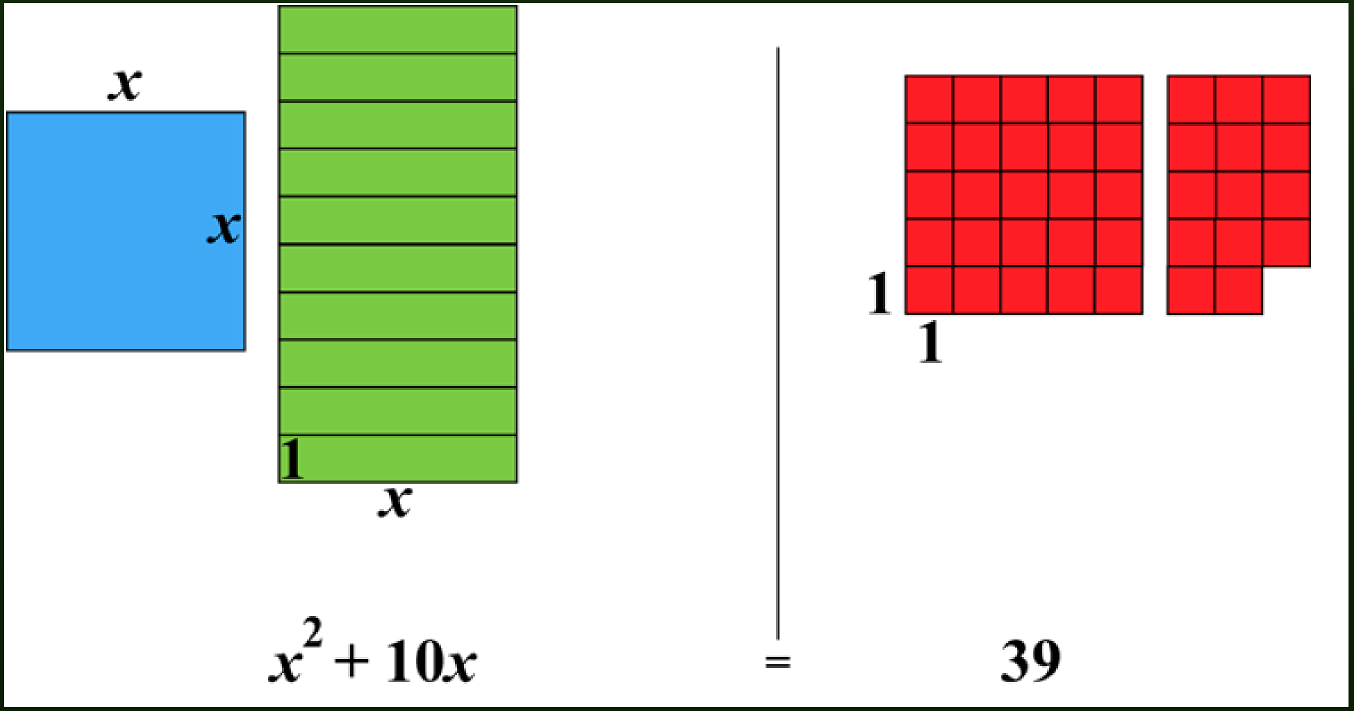
This problem can be written algebraically:

$$x^2 + 10x = 39$$

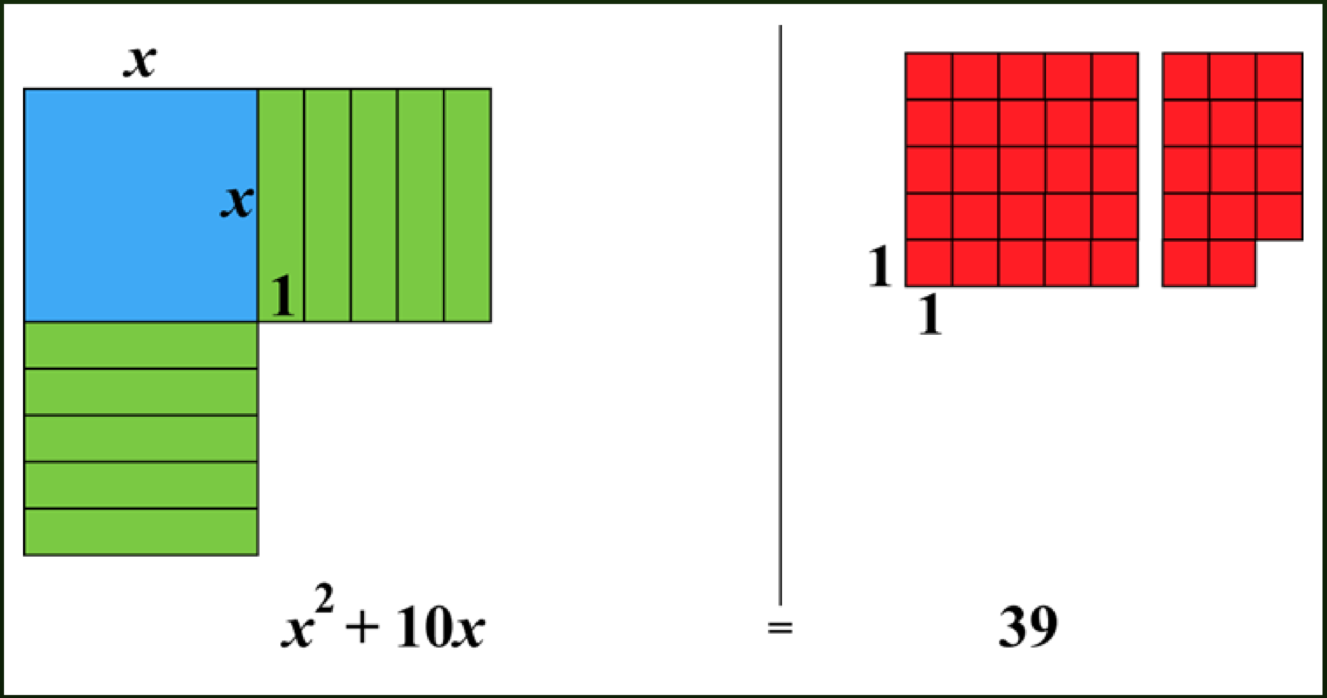
How would you solve it?

Completing the Square

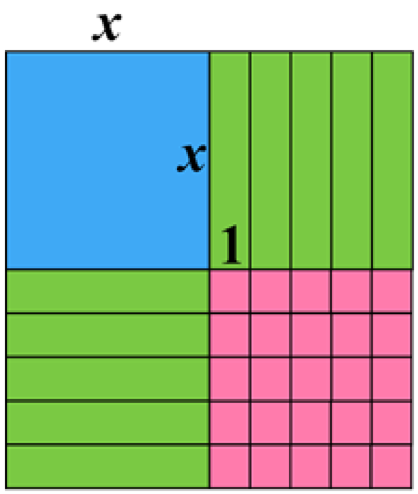
Solving with Algebra Tiles:

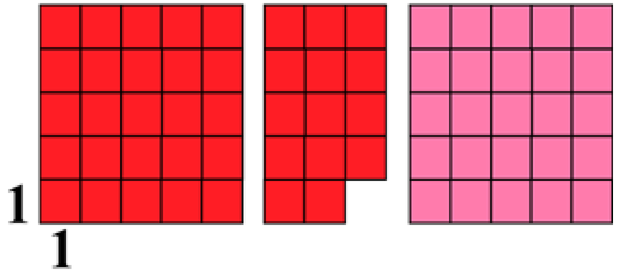


Completing the Square



Completing the Square





$$x^2 + 10x + 25 = 39 + 25$$

$$(x + 5)^2 = 64$$

$$x = \{3, -13\}$$

Completing the Square

Traditional algebraic approach for completing the square:

Scrap:

$$\frac{10}{2} = 5$$

$$5^2 = 25$$

Calculations:

$$x^2 + 10x = 39$$

$$x^2 + 10x + 25 = 39 + 25$$

$$(x+5)^2 = 64$$

$$x + 5 = \pm 8$$

$$x = \{3, -13\}$$

Quadratic Formula

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Completing the Square

The solution written:

- You half the number of roots, which in the present instance yields five.
- This you multiply by itself; the product is twenty-five. Add this to thirty-nine; the sum is sixty-four.
- Now take the root of this, which is eight, and subtract from it half the number of roots, which is five; the remainder is three.
- This is the root of the square which you sought for; the square itself is nine.

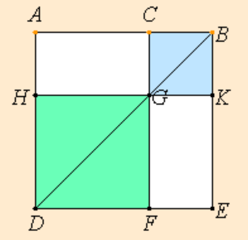
Quadratics

Euclid's Elements
Book II
Proposition 4

If a straight line is cut at random, then the square on the whole equals the sum of the squares on the segments plus twice the rectangle contained by the segments.

Let the straight line *AB* be cut at random at *C*.

I say that the square on *AB* equals the sum of the squares on *AC* and *CB* plus twice the rectangle *AC* by *CB*.



Describe the square *ADEB* on *AB*. Join *BD*. Draw *CF* through *C* parallel to either *AD* or *EB*, and draw *HK* through *G* parallel to either *AB* or *DE*. I.46 I.31

Then, since *CF* is parallel to *AD*, and *BD* falls on them, the exterior angle *CGB* equals the interior and opposite angle *ADB*. I.29

But the angle *ADB* equals the angle *ABD*, since the side *BA* also equals *AD*. Therefore the angle *CGB* also equals the angle *GBC*, so that the side *BC* also equals the side *CG*. I.5 I.6

But *CB* equals *GK*, and *CG* to *KB*. Therefore *GK* also equals *KB*. Therefore *CGKB* is equilateral. I.34

I say next that it is also right-angled.

Since *CG* is parallel to *BK*, the sum of the angles *KBC* and *GCB* equals two right angles. I.29

But the angle *KBC* is right. Therefore the angle *BCG* is also right, so that the opposite angles *CGK* and *GKB* are also right. I.34

Therefore *CGKB* is right-angled, and it was also proved equilateral, therefore it is a square, and it is described on *CB*.

For the same reason *HF* is also a square, and it is described on *HG*, that is *AC*. Therefore the squares *HF* and *KC* are the squares on *AC* and *CB*. I.34

Now, since *AG* equals *GE*, and *AG* is the rectangle *AC* by *CB*, for *GC* equals *CB*, therefore *GE* also equals the rectangle *AC* by *CB*. Therefore the sum of *AG* and *GE* equals twice the rectangle *AC* by *CB*. I.43

But the squares *HF* and *CK* are also the squares on *AC* and *CB*, therefore the sum of the four figures *HF*, *CK*, *AG*, and *GE* equals the sum of the squares on *AC* and *CB* plus twice the rectangle *AC* by *CB*.

But *HF*, *CK*, *AG*, and *GE* are the whole *ADEB*, which is the square on *AB*.

Therefore the square on *AB* equals the the sum of the squares on *AC* and *CB* plus twice the rectangle *AC* by *CB*.

David Joyce Clark University

Quadratics

If a straight line is cut at random, then the square on the whole equals the sum of the squares on the segments plus twice the rectangle contained by the segments.

Quadratics

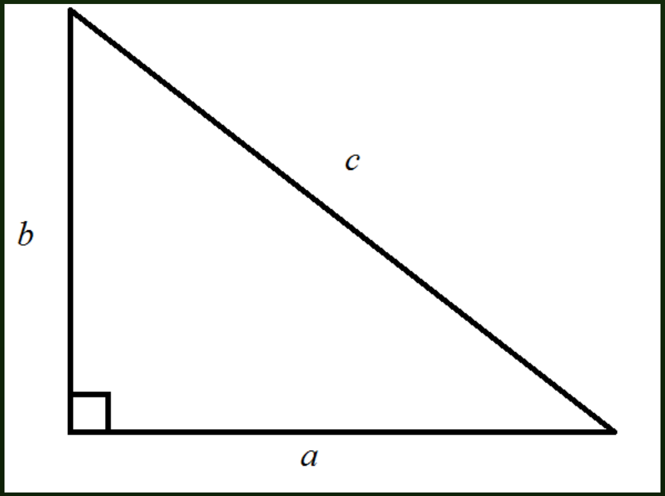
A geometric proof:

xplusasq 2.swf

Pythagorean Theorem

For a right triangle with legs a and b, and hypotenuse c

$$a^2 + b^2 = c^2$$



Pythagorean Theorem

If we require that the sides of the triangles are positive integers, then we are dealing with Pythagorean Triples.

How many can you think of?

Pythagorean Triples

3, 4, 5

5, 12, 13

8, 15, 17

Others?

Pythagorean Triples

Find a Pythagorean triple in which either a or b is 48.

36, 48, 60   20, 48, 52   48, 90,102

There are 10 triples. How do we find them?

Is it just trial and error or is there a systematic method we might use?

Pythagorean Triples

If you pick two positive integers

u and v

with

$$u > v$$

then

$$a = u^2 - v^2$$

$$b = 2uv$$

$$c = u^2 + v^2$$

will form a Pythagorean triple. This can be easily verified by doing the algebra.

Pythagorean Triples

For example:

If

$$u = 7 \text{ and } v = 4$$

then

$$a = 49 - 16 = 33$$

$$b = 2(7)(4) = 56$$

$$c = 49 + 16 = 65$$

and 33, 56, 65 is a Pythagorean triple

Pythagorean Triples

To return to the problem of finding a right triangle with leg 48, we might observe :

If

$$2uv = 48$$

then we might choose

$$u = 8 \text{ and } v = 3$$

so

$$a = 64 - 9 = 55$$

$$b = 2(8)(3) = 48$$

$$c = 64 + 9 = 73$$

55, 48, 73 is a Pythagorean triple.

Pythagorean Triples

If you wanted $u^2 - v^2 = 48$

Consider $u^2 - v^2 = (u - v)(u + v)$

If we choose

$$u + v = 8 \text{ and } u - v = 6$$

Solving these simultaneous equations yields

$$u = 7 \text{ and } v = 1$$

Then

$$a = 49 - 1 = 48$$

$$b = 2(7)(1) = 14$$

$$c = 49 + 1 = 50$$

48, 14, 50 is a Pythagorean Triple

Pythagorean Triples

Find a Pythagorean triple where c is a perfect square.

Again looking at the triples we have already generated you could easily come up with 15, 20, 25 and 65, 156, 169.

In both cases c is a perfect square.

However if you try to generate a, b, and c, using u and v, you will find that you can't do it.

When a, b, and c have a common factor > 1, the formulas we used before won't work without modification.

Pythagorean Triples

So let us ask ourselves if it is possible to find u and v so that there is a Pythagorean triple with $c = 25$.

$$16 + 9 = 25$$

Thus

$$u = 4 \text{ and } v = 3$$

Thus

$$a = 16 - 9 = 7$$

$$b = 2(4)(3) = 24$$

Likewise $169 = 144 + 25$ so $u = 12$ and $v = 5$ so that $a = 119$ and $b = 120$.

Thus 7, 24, 25 and 119, 120, 169 are Pythagorean triples with c equal to a perfect square.

Pythagorean Triples

A challenging problem is to determine which perfect squares can be equal to c in a Pythagorean triple.

Another interesting problem based on the Pythagorean triples 3, 4, 5 and 119, 120, 169 is to determine Pythagorean triples where a and b differ by one.

Plimpton 322

Plimpton 322 is a Babylonian clay tablet dating back to approximately 1800 BCE. Hundreds of thousands of these tablets have been found The discovery and decipherment of those tablets which might contain mathematical material is an exciting chapter in the history of mathematics. We shall exam Plimpton 322 briefly in order to give a flavor of the work necessary to understand a piece of Babylonian mathematics.

The material that follows is based on the article Buck,R.C. "Sherlock Holmes in Babylon," Amer. Math. Monthly 87(1980).

Plimpton 322

Plimpton 322

B	C	(a, b)
119	169	12,5
3367	11521	?
4601	6649	75,32
12709	18541	125,54
65	97	9, 4
319	481	20,9
2291	3541	54,25
799	1249	32,15
541	769	?
4961	8161	81,40
45	75	?
1679	2929	48,25
25921	289	?
1771	3229	50,27
56	53	?

Figure 9.

B	C	(a, b)
119	169	12, 5
3367	4825	64, 27
4601	6649	75, 32
12709	1854	125, 54
65	97	9, 4
319	481	20, 9
2291	3541	54, 25
799	1249	32, 15
481	769	25, 12
4961	8161	81, 40
45	75	$1, \frac{1}{2} = 30$
1679	2929	48, 25
161	289	15, 8
1771	3229	50, 27
56	106	9, 5

Figure 10. Corrected Version

Plimpton 322

The two numbers in the first row are hopefully familiar **119** **169** We just showed that 119, 120, 169 form a Pythagorean triple.

If we search the rest of the numbers in these two columns we find that 10 of the 15 pairs of numbers are actually two parts of a Pythagorean triple. In addition we can find plausible reasons why the other numbers were also meant to be Pythagorean triples but this is beyond the scope of this talk.

Plimpton 322

The notation used in the text of the article lets C equal the hypotenuse of triangle, B the leg of the triangle given on the tablet and lets D be the implied third side. The following diagram is helpful for the next part of the discussion

Plimpton 322

The first column of the tablet contain the squares of B/D Focusing just on the ratio B/D for the moment, I would like to look at two aspects of this ratio. Based on the diagram B/D is the tangent of angle theta.

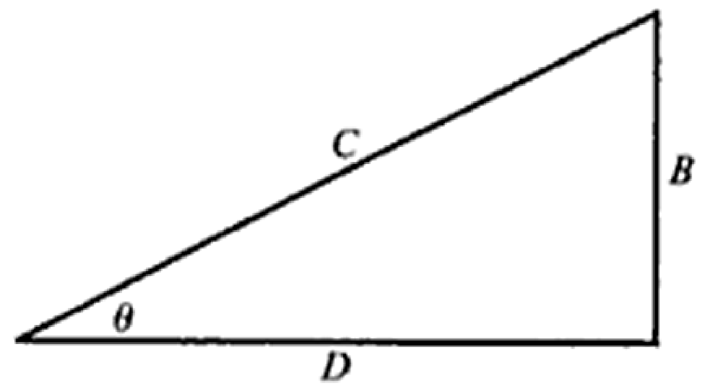


Figure 11. $B = a^2 - b^2, D = 2ab, C = a^2 + b^2$

Plimpton 322

For the triangle 119,120, 169, the two legs are almost the same.

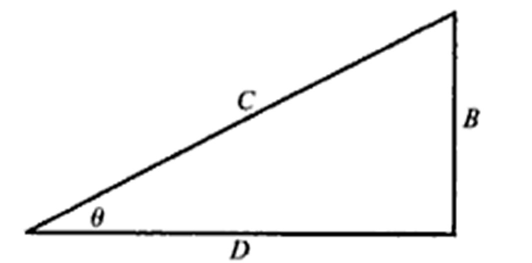


Figure 11. $B = a^2 - b^2, D = 2ab, C = a^2 + b^2$

We could look at it as almost an isosceles right triangle. If it were an isosceles right triangle then theta would be 45 degrees.

If you were to calculate D for each of the B and C values in the table and then calculate B/D an interesting pattern emerges.

Plimpton 322

B/D is tangent theta.

If you take the inverse tangent of *B/D* you get a series of angles which decrease in an orderly fashion down to approximately 30 degrees.

Are we looking at an example of a trig table?

Plimpton 322

Unfortunately my time is up.

When I taught History of Mathematics courses, I told my students that I would tell them lots of stories some of which might be true. This is a fascinating story.

To read another interpretation see the article by Robson in the bibliography.

10 Pythagorean triangles with one leg equal to 48

14, 48, 50	20, 48, 52
36, 48, 60	55, 48, 73
64, 48, 80	90, 48, 102
140, 48, 148	189, 48, 195
286, 48, 290	576, 48, 577

Resources

Beiler, A.H. (1966) *Recreations in the Theory of Numbers.* Dover, New York, p. 111.

Berlinghoff, William P., Gouvêa, Fernando Quadros, *Math through the Ages: A Gentle History for Teachers and Others.* © 2004. Mathematical Association of America and Oxton House Publishers. p. 127.

Buck, R, Creighton (1980), "Sherlock Holmes in Babylon", *American Mathematical Monthly*, Mathematical Association of America, 87 (5): 335-345.

Netz, Reviel and Noel, William. *The Archimedes Codex Revealing the Secrets of the World's Greatest Palimpsest*. Weidenfeld & Nicolson, London 2007.

Robson, Eleanor (2001) "Neither Sherlock Holmes nor Babylon: A reassessment of Plimpton 322", *Historia Math*. 28(3):167-206.

Resources

http://www.corestandards.org/the-standards/mathematics/grade-8/geometry/

http://aleph0.clarku.edu/~djoyce/java/elements/bookII/propII4.html

http://aleph0.clarku.edu/~djoyce/java/elements/bookIX/propIX36.html

http://aleph0.clarku.edu/~djoyce/mathhist/plimpnote.html

http://www-history.mcs.st-and.ac.uk/Biographies/Al-Khwarizmi.html

http://www-history.mcs.st-and.ac.uk/PictDisplay/Archimedes.html

http://www-history.mcs.st-and.ac.uk/PictDisplay/Euclid.html

http://nlvm.usu.edu/en/nav/frames_asid_189_g_3_t_2.html?open=activities&from=topic_t_2.html

Resources

http://www.math.ubc.ca/~cass/Euclid/book1/byrne-48.html

http://nlvm.usu.edu/en/nav/frames_asid_189_g_3_t_2.html?open=activities&from=topic_t_2.html

http://jeff560.tripod.com/

http://www.math.cornell.edu/~mec/GeometricDissections/pictures/Stomachion.JPG

http://www.math.cornell.edu/~mec/GeometricDissections/1.2%20Archimedes%20Stomachion.html

http://www.nytimes.com/2003/12/14/us/in-archimedes-puzzle-a-new-eureka-moment.html?pagewanted=2&src=pm

http://archimedespalimpsest.org/scholarship_netz3.html